Supplementary material for "Computer-assisted global analysis for vibro-impact dynamics: a reduced smooth maps approach"

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I. EXACT MAP DERIVATIONS

Section 2 of the main text summarizes the results in [2] for calculating the exact maps between two consecutive impacts. Ideally, we would like to derive closed-form solutions at the $(j+1)^{\text{th}}$ impact $(\dot{Z}_{j+1}, t_{j+1})$ if given the j^{th} impact. The closed-form solutions enable us to study the global dynamics of the VI-EH system. However, we briefly showed in Section 2 of the main text that deriving the closed-form solutions is not feasible. In this section, we expand the solving process for the closed-form analytical expressions for the BTB motion.

BTB motion is composed of maps $P_{TB} \circ P_{BT}$, where map $P_{TB} : (Z_j \in \partial T, \dot{Z}_j, t_j) \mapsto (Z_{j+1} \in \partial B, \dot{Z}_{j+1}, t_{j+1})$ and map $P_{BT} : (Z_j \in \partial B, \dot{Z}_j, t_j) \mapsto (Z_{j+1} \in \partial T, \dot{Z}_{j+1}, t_{j+1})$. The equations are

$$\begin{aligned} P_{BT}: \quad & \dot{Z}_{j+1} = -r\dot{Z}_j + \bar{g}\cdot(t_{j+1}-t_j) + F_1(t_{j+1}) - F_1(t_j), \\ & -\frac{d}{2} = \frac{d}{2} - r\dot{Z}_j\cdot(t_{j+1}-t_j) + \frac{\bar{g}}{2}\cdot(t_{j+1}-t_j)^2 + F_2(t_{j+1}) - F_2(t_j) - F_1(t_j)\cdot(t_{j+1}-t_j) \end{aligned} \tag{S.1}$$

$$\begin{aligned} P_{TB}: \quad & \dot{Z}_{j+2} = -r\dot{Z}_{j+1} + \bar{g}\cdot(t_{j+2}-t_{j+1}) + F_1(t_{j+2}) - F_1(t_{j+1}), \\ & \frac{d}{2} = -\frac{d}{2} - r\dot{Z}_{j+1}\cdot(t_{j+2}-t_{j+1}) + \frac{\bar{g}}{2}\cdot(t_{j+2}-t_{j+1})^2 + F_2(t_{j+2}) - F_2(t_{j+1}) - F_1(t_{j+1})\cdot(t_{j+2}-t_{j+1}) \end{aligned} \tag{S.2}$$

If given $f(t) = F(\pi t + \varphi) = \cos(\pi t + \varphi)$, then $F_1(t) = \frac{1}{\pi}\sin(\pi t + \varphi)$ and $F_2(t) = -\frac{1}{\pi^2}\cos(\pi t + \varphi)$. The maps P_{BT}, P_{TB} can be written as follows:

$$\dot{Z}_{j+1} = -r\dot{Z}_j + \bar{g}\cdot(t_{j+1}-t_j) + \frac{1}{\pi}\Big(\sin(\pi t_{j+1}+\varphi) - \sin(\pi t_j+\varphi)\Big), \tag{S.3}$$

$$\begin{aligned} -d = & -r\dot{Z}_j\cdot(t_{j+1}-t_j) + \frac{\bar{g}}{2}\cdot(t_{j+1}-t_j)^2 - \frac{1}{\pi^2}\Big(\cos(\pi t_{j+1}+\varphi) - \cos(\pi t_j+\varphi)\Big) \\ & - \frac{1}{\pi}(t_{j+1}-t_j)\sin(\pi t_j+\varphi), \end{aligned} \tag{S.4}$$

$$\dot{Z}_{j+2} = -r\dot{Z}_{j+1} + \bar{g}\cdot(t_{j+2}-t_{j+1}) + \frac{1}{\pi}\Big(\sin(\pi t_{j+2}+\varphi) - \sin(\pi t_{j+1}+\varphi)\Big) \tag{S.5}$$

$$\begin{aligned} d = & -r\dot{Z}_{j+1}\cdot(t_{j+2}-t_{j+1}) + \frac{\bar{g}}{2}\cdot(t_{j+2}-t_{j+1})^2 - \frac{1}{\pi^2}\Big(\cos(\pi t_{j+2}+\varphi) - \cos(\pi t_{j+1}+\varphi)\Big) \\ & - \frac{1}{\pi}(t_{j+2}-t_{j+1})\sin(\pi t_{j+1}+\varphi) \end{aligned} \tag{S.6}$$

First, unpack (S.4):

$$\begin{aligned} -d = & -r\dot{Z}_j t_{j+1} + r\dot{Z}_j t_j + \frac{\bar{g}}{2}t_{j+1}^2 - \bar{g}t_{j+1}t_j + \frac{\bar{g}}{2}t_j^2 - \frac{1}{\pi^2}\cos(\pi t_{j+1}+\varphi) + \frac{1}{\pi^2}\cos(\pi t_j+\varphi) \\ & - \frac{1}{\pi}\sin(\pi t_j+\varphi)t_{j+1} + \frac{1}{\pi}\sin(\pi t_j+\varphi)t_j \end{aligned}$$

Sort all terms containing t_{j+1} such that it's a quadratic equation on the LHS and cosine on the RHS:

$$\begin{aligned} & \frac{\bar{g}}{2}t_{j+1}^2 - \Big(r\dot{Z}_j + \bar{g}t_j + \frac{1}{\pi}\sin(\pi t_j+\varphi)\Big)t_{j+1} + \Big(d + r\dot{Z}_j t_j + \frac{\bar{g}}{2}t_j^2 + \frac{1}{\pi^2}\cos(\pi t_j+\varphi) + \frac{t_j}{\pi}\sin(\pi t_j+\varphi)\Big) \\ & = \frac{1}{\pi^2}\cos(\pi t_{j+1}+\varphi) \end{aligned} \tag{S.7}$$

Equation (S.7) has a solution if the quadratic function on the LHS and the cosine function on the RHS have an intersection. If t_{j+1} has a closed-form solution, then the expression can be written as a function of $\dot{Z}_j$ and t_j:

$$t_{j+1} = h_1(\dot{Z}_j, t_j) \tag{S.8}$$

Observe equation (S.3) that $\dot{Z}_{j+1}$ is a function of $\dot{Z}_j, t_{j+1}$, and t_j. Apply (S.8) in (S.3), we can rewrite (S.3) as a function of $\dot{Z}_j$ and t_j only:

$$\begin{aligned}\dot{Z}_{j+1} &= h_2(\dot{Z}_j, t_{j+1}, t_j)\\ &= h_2(\dot{Z}_j, h_1(\dot{Z}_j, t_j), t_j)\\ &= h_3(\dot{Z}_j, t_j)\end{aligned} \tag{S.9}$$

By unpacking and regrouping (S.6), the equation can be written as a quadratic equation of t_{j+2} on the LHS and cosine function of t_{j+2} on the RHS as well. This means t_{j+2} can be written as a function of $\dot{Z}_{j+1}$ and t_{j+1}. Apply (S.8) and (S.9), we have

$$\begin{aligned}t_{j+2} &= h_4(\dot{Z}_{j+1}, t_{j+1})\\ &= h_4(h_3(\dot{Z}_j, t_j), h_1(\dot{Z}_j, t_j))\\ &= h_5(\dot{Z}_j, t_j)\end{aligned} \tag{S.10}$$

by definition of ψ_j, $\psi_{j+2} = \mod(\pi t_{j+2} + \varphi, 2\pi) = h_6(\dot{Z}_j, t_j)$.

Applying (S.8), (S.9), (S.10) into (S.5), $\dot{Z}_{j+2}$ can be written as a function of $\dot{Z}_j, t_j$ as well:

$$\begin{aligned}\dot{Z}_{j+2} &= h_7(\dot{Z}_{j+1}, t_{j+2}, t_{j+1})\\ &= h_7(h_3(\dot{Z}_j, t_j), h_5(\dot{Z}_j, t_j), h_1(\dot{Z}_j, t_j))\\ &= h_8(\dot{Z}_j, t_j)\end{aligned}$$

Therefore, the exact map for BTB case $P_{TB} \circ P_{BT}$ can be written as

$$\begin{aligned}\dot{Z}_{j+2} &= h_8(\dot{Z}_j, t_j)\\ \psi_{j+2} &= h_6(\dot{Z}_j, t_j)\end{aligned}$$

Solving (S.7) which involves finding t_{j+1} that satisfies both the quadratic equation on the LHS and the cosine function on the RHS is the main obstacle in finding the closed form solution for $(\dot{Z}_{j+2}, t_{j+2})$. If we had the explicit expression for $t_{j+1} = h_1(\dot{Z}_j, t_j)$, the explicit expressions for $h_i(\dot{Z}_j, t_j), i = 2, 3, ..., 8$ would follow. However, it is not possible to write down an explicit expression for the solution to (S.7). Therefore, we are not able to find the closed form expressions for $\dot{Z}_{j+2}$ and ψ_{j+2}.

II. COEFFICIENTS FOR THE COMPOSITE MAP

In Section 4 of the main text, an algorithm for constructing the composite map is developed. The composite map combines the approximate return maps for each subregion $\mathcal{R}_i$ for $i = 1, 2, 3, 4, 5$ in Fig. 3(b) of the main text. The approximate return maps are given in terms of the variables (v_k, ϕ_k) that denote the approximate relative impact velocity on ∂B and the corresponding impact phase, respectively, at the k^{th} return to ∂B. In this section, we give the specific coefficients of the approximate maps.

A. Region $\mathcal{R}_1$

The polynomial approximate map of $\mathcal{R}_1$:

$$\begin{aligned}g_1(v_k, \phi_k) &= a_0 + a_1\phi_k + a_2 v_k + a_3\phi_k^2 + a_4\phi_k v_k + a_5 v_k^2 + a_6\phi_k^2 v_k + a_7\phi_k v_k^2 + a_8 v_k^3,\\ f_1(v_k, \phi_k) &= b_0 + b_1\phi_k + b_2 v_k + b_3\phi_k^2 + b_4\phi_k v_k + b_5 v_k^2 + b_6\phi_k^2 v_k + b_7\phi_k v_k^2 + b_8 v_k^3,\end{aligned} \tag{S.11}$$

where the coefficients are functions of d.

$$
\begin{aligned}
a_0 &= -1.499d^2 + 18.39d + 10.21, & b_0 &= -381.7d^2 + 210d - 27.04,\\
a_1 &= -196.5d^2 + 146.2d - 51.59, & b_1 &= 777.6d^2 - 438.9d + 59.89,\\
a_2 &= 81.56d^2 - 47.97d - 28.93, & b_2 &= 1036d^2 - 567.3d + 76.38,\\
a_3 &= 257.1d^2 - 189.4d + 45.34, & b_3 &= -487.7d^2 + 278.1d - 38.23,\\
a_4 &= 380.3d^2 - 321.8d + 104.5, & b_4 &= -1504d^2 + 860.1d - 121,\\
a_5 &= -218.2d^2 + 125.2d + 7.025, & b_5 &= -961.4d^2 + 535.5d - 75.31,\\
a_6 &= -361d^2 + 268.6d - 59.56, & b_6 &= 599.4d^2 - 345.7d + 48.48,\\
a_7 &= -84.22d^2 + 91.69d - 35.86, & b_7 &= 706.9d^2 - 413.1d + 60.34,\\
a_8 &= 167.4d^2 - 111.7d + 14.11, & b_8 &= 313.2d^2 - 180d + 26.68.
\end{aligned}
$$

B. Region $\mathcal{R}_2$

The polynomial approximate map of $\mathcal{R}_2$:

$$
\begin{aligned}
g_2(\phi_k) &= a_{20}\phi_k^5 + a_{21}\phi_k^4 + a_{22}\phi_k^3 + a_{23}\phi_k^2 + a_{24}\phi_k + a_{25},\\
f_2(v_k) &= b_{20}v_k^5 + b_{21}v_k^4 + b_{22}v_k^3 + b_{23}v_k^2 + b_{24}v_k + b_{25},
\end{aligned}
\tag{S.12}
$$

where the coefficients are functions of d.

$$
\begin{aligned}
a_{20} &= -314721.3d^5 + 491841.99d^4 - 306600.36d^3 + 95280.8d^2 - 14757.75d + 910.99,\\
a_{21} &= 3254508.65d^5 - 5091024.58d^4 + 3176650.07d^3 - 988128.92d^2 + 153191.85d - 9465.61,\\
a_{22} &= -12716817.41d^5 + 19914833.37d^4 - 12439815.01d^3 + 3873678.24d^2 - 601181.49d + 37187.01,\\
a_{23} &= 23067186.96d^5 - 36172597.397d^4 + 22625380.93d^3 - 7054633.67d^2 + 1096273.23d - 67901.79,\\
a_{24} &= -18352978.56d^5 + 28821687.94d^4 - 18053223.19d^3 + 5636927.09d^2 - 877183.51d + 54409.05,\\
a_{25} &= 4085295.603d^5 - 6431417.18d^4 + 4038349.16d^3 - 1263982.02d^2 + 197165.56d - 12259.01,
\end{aligned}
$$

$$
\begin{aligned}
b_{20} &= -8423791.87d^4 + 10162592.6d^3 - 4551825.9d^2 + 895903.8d - 65176,\\
b_{21} &= 39053115.4d^4 - 47089789.2d^3 + 21089709.9d^2 - 4152787.8d + 302441.7,\\
b_{22} &= -72167960.4d^4 + 86937329.1d^3 - 38914922.6d^2 + 7662387.6d - 558347.5,\\
b_{23} &= 66515161.4d^4 - 80025768.3d^3 + 35789133.6d^2 - 7043852.7d + 513345.19,\\
b_{24} &= -30587995.6d^4 + 36746130.9d^3 - 16414881.8d^2 + 3228420.6d - 235247.1,\\
b_{25} &= 5609812.9d^4 - 6728373.2d^3 + 3001779.6d^2 - 589859.9d + 42967.7.
\end{aligned}
$$

C. Region $\mathcal{R}_3$

The polynomial approximate map of $\mathcal{R}_3$:

$$
\begin{aligned}
g_3(v_k,\phi_k) &= a_{300} + a_{301}\phi_k + a_{302}v_k + a_{303}\phi_k^2 + a_{304}\phi_k v_k + a_{305}v_k^2 + a_{306}\phi_k^3 + a_{307}\phi_k^2 v_k + a_{308}\phi_k v_k^2\\
&\quad + a_{309}v_k^3 + a_{310}\phi_k^4 + a_{311}\phi_k^3 v_k + a_{312}\phi_k^2 v_k^2 + a_{313}\phi_k v_k^3 + a_{314}v_k^4 + a_{315}\phi_k^4 v_k\\
&\quad + a_{316}\phi_k^3 v_k^2 + a_{317}\phi_k^2 v_k^3 + a_{318}\phi_k v_k^4 + a_{319}v_k^5,\\
f_3(v_k,\phi_k) &= b_{300} + b_{301}\phi_k + b_{302}v_k + b_{303}\phi_k^2 + b_{304}\phi_k v_k + b_{305}v_k^2 + b_{306}\phi_k^3 + b_{307}\phi_k^2 v_k + b_{308}\phi_k v_k^2 + b_{309}v_k^3\\
&\quad + b_{310}\phi_k^3 v_k + b_{311}\phi_k^2 v_k^2 + b_{312}\phi_k v_k^3 + b_{313}v_k^4 + b_{314}\phi_k^3 v_k^2 + b_{315}\phi_k^2 v_k^3 + b_{316}\phi_k v_k^4 + b_{317}v_k^5,
\end{aligned}
\tag{S.13}
$$

where the coefficients are constants.

$$
\begin{aligned}
&a_{300} = -4.708 \cdot 10^{-5}, && a_{310} = 0.02214, && b_{300} = 3.311 \cdot 10^{-5}, && b_{310} = 0.09745,\\
&a_{301} = 0.99, && a_{311} = -45.86, && b_{301} = 0.0002375, && b_{311} = 16.8,\\
&a_{302} = 3.456, && a_{312} = -235.6, && b_{302} = 0.4358, && b_{312} = 44.31,\\
&a_{303} = 0.0483, && a_{313} = -323.4, && b_{303} = -0.0001751, && b_{313} = 29.58,\\
&a_{304} = -11.35, && a_{314} = -148, && b_{304} = 0.268, && b_{314} = -8.853,\\
&a_{305} = -13.29, && a_{315} = 18.32, && b_{305} = 1.895, && b_{315} = -38.48,\\
&a_{306} = -0.06063, && a_{316} = 143, && b_{306} = 8.499 \cdot 10^{-6}, && b_{316} = -51.93,\\
&a_{307} = 39.33, && a_{317} = 331.9, && b_{307} = -0.3043, && b_{317} = -24.49,\\
&a_{308} = 111.7, && a_{318} = 308.4, && b_{308} = -10.54,\\
&a_{309} = 70.26, && a_{319} = 114.2, && b_{309} = -12.81.
\end{aligned}
$$

D. Region $\mathcal{R}_4$

The polynomial approximate map of $\mathcal{R}_4$:

$$
\begin{aligned}
g_4(\phi_k) &= a_{40}\phi_k^4 + a_{41}\phi_k^3 + a_{42}\phi_k^2 + a_{43}\phi_k + a_{44},\\
f_4(v_k) &= b_{40}v_k^8 + b_{41}v_k^7 + b_{42}v_k^6 + b_{43}v_k^5 + b_{44}v_k^4 + b_{45}v_k^3 + b_{46}v_k^2 + b_{47}v_k + b_{48},
\end{aligned}
\tag{S.14}
$$

where the coefficients are functions of d.

$$
\begin{aligned}
a_{40} &= -25564661d^5 + 38856593d^4 - 23532532d^3 + 7099885d^2 - 1067289d + 63961,\\
a_{41} &= 187346514d^5 - 284624988d^4 + 172304032d^3 - 51964934d^2 + 7808829d - 467815,\\
a_{42} &= -508479594d^5 + 772240827d^4 - 467346559d^3 + 140905675d^2 - 21168395d + 1267853,\\
a_{43} &= 605074088d^5 - 918738962d^4 + 555892718d^3 - 167571538d^2 + 25170155d - 1507297,\\
a_{44} &= -267117434d^5 + 405554166d^4 - 245366553d^3 + 73959909d^2 - 11108535d + 665192,
\end{aligned}
$$

$$
\begin{aligned}
b_{40} &= -33678323446d^4 + 39732483684d^3 - 17535685854d^2 + 3431234055d - 251148526,\\
b_{41} &= 83698133214d^4 - 98744923307d^3 + 43580936553d^2 - 8527653549d + 624188381,\\
b_{42} &= -87552753895d^4 + 103292995807d^3 - 45588589509d^2 + 8920591453d - 652957616,\\
b_{43} &= 50107657144d^4 - 59115843570d^3 + 26090961385d^2 - 5105403132d + 373701618,\\
b_{44} &= -17068153916d^4 + 20136255271d^3 - 8887112566d^2 + 1738997288d - 127289906,\\
b_{45} &= 3522641275d^4 - 4155665872d^3 + 1834037891d^2 - 358869361d + 26267846,\\
b_{46} &= -427643189d^4 + 504439390d^3 - 222608632d^2 + 43555467d - 3187931,\\
b_{47} &= 27780980d^4 - 32762498d^3 + 14455481d^2 - 2827938d + 206958,\\
b_{48} &= -737999d^4 + 869871d^3 - 383650d^2 + 75032d - 5489.
\end{aligned}
$$

E. Region $\mathcal{R}_5$

The polynomial approximate map of $\mathcal{R}_5$:

$$
\begin{aligned}
g_5(\phi_k) &= a_{50}\phi_k^3 + a_{51}\phi_k^2 + a_{52}\phi_k + a_{53},\\
f_5(v_k) &= |b_{50}v_k^4 + b_{51}v_k^3 + b_{52}v_k^2 + b_{53}v_k + b_{54}|,
\end{aligned}
\tag{S.15}
$$

where the coefficients are functions of d.

$$
\begin{aligned}
a_{50} &= 2064.98d^4 - 1231.18d^3 - 75.3328d^2 + 138.871d - 19.476, \\
a_{51} &= -19752.202d^4 + 12355.348d^3 + 119.244d^2 - 1133.696d + 166.629, \\
a_{52} &= 61428.79d^4 - 39362.33d^3 + 662.836d^2 + 3177.32d - 485.139, \\
a_{53} &= -62366.62d^4 + 40245.2d^3 - 1078.83d^2 - 3068.1d + 482.49,
\end{aligned}
$$

$$
\begin{aligned}
b_{50} &= -3327935009d^4 + 4251589868d^3 - 2036587076d^2 + 433686951d - 34659098, \\
b_{51} &= 49128168d^4 - 628668996d^3 + 301980243d^2 - 64578564d + 5193379, \\
b_{52} &= -24532591d^4 + 31293322d^3 - 15007193d^2 + 3211068d - 259329, \\
b_{53} &= 438110d^4 - 552384d^3 + 262235d^2 - 55690.3d + 4496.38, \\
b_{54} &= -5.8882d^4 + 7.2206d^3 - 3.2965d^2 + 0.6646d - 0.0499.
\end{aligned}
$$

III. THE COEFFICIENTS FOR THE v SECOND-ITERATE MAP

In Section 6 of the main text, a method using the auxiliary map approach is developed. In regions where the approximate maps are two-dimensional, we decouple the 2D systems into the 1D auxiliary maps. We construct a new composite map $\mathcal{M}_{\mathcal{A}}^{(N)}$, defined in (6.3), which assists us in identifying the global attracting region of the system. In the meantime, a higher-order iterate map is derived using $\mathcal{M}_{\mathcal{A}}^{(N)}$ to show the global dynamics and to pinpoint the location of the global absorbing domain.

In the case of v_k, the second iterate map, equation (6.10) in the main text, has a closed-form expression:

$$
\begin{aligned}
v_{k+2}(v_k; \phi_{\min}, \phi_{\max}) &= f_n(f_n(v_k, \phi_{\max}), \phi_{\min}) \\
&= \alpha_0 + \alpha_1 v_k^1 + \alpha_2 v_k^2 + \alpha_3 v_k^3 + \alpha_4 v_k^4 + \alpha_5 v_k^5 + \alpha_6 v_k^6 + \alpha_7 v_k^7 + \alpha_8 v_k^8 + \alpha_9 v_k^9,
\end{aligned} \tag{S.16}
$$

where $\alpha_i, i = 1, ..., 9$ can be calculated if given the parameters $d, \phi_{\min}, \phi_{\max}$ since $b_1, ..., b_8$ are functions of d. Each coefficient α_i are polynomials with combinations $b_0, b_1, ..., b_9$, $\phi_{\min}$, and $\phi_{\max}$:

$$
\begin{aligned}
\alpha_0 = b_0 &+ b_0 b_2 + b_0^2 b_5 + b_0^3 b_8 + b_1 \phi_{\min} + b_0 b_4 \phi_{\min} + b_0^2 b_7 \phi_{\min} + b_3 \phi_{\min}^2 + b_0 b_6 \phi_{\min}^2 + b_1 b_2 \phi_{\max} + 2 b_0 b_1 b_5 \phi_{\max} \\
&+ 3 b_0^2 b_1 b_8 \phi_{\max} + b_1 b_4 \phi_{\min} \phi_{\max} + 2 b_0 b_1 b_7 \phi_{\min} \phi_{\max} + b_1 b_6 \phi_{\min}^2 \phi_{\max} + b_2 b_3 \phi_{\max}^2 + b_1^2 b_5 \phi_{\max}^2 + 2 b_0 b_3 b_5 \phi_{\max}^2 \\
&+ 3 b_0 b_1^2 b_8 \phi_{\max}^2 + 3 b_0^2 b_3 b_8 \phi_{\max}^2 + b_3 b_4 \phi_{\min} \phi_{\max}^2 + b_1^2 b_7 \phi_{\min} \phi_{\max}^2 + 2 b_0 b_3 b_7 \phi_{\min} \phi_{\max}^2 + b_3 b_6 \phi_{\min}^2 \phi_{\max}^2 \\
&+ 2 b_1 b_3 b_5 \phi_{\max}^3 + b_1^3 b_8 \phi_{\max}^3 + 6 b_0 b_1 b_3 b_8 \phi_{\max}^3 + 2 b_1 b_3 b_7 \phi_{\min} \phi_{\max}^3 + b_3^2 b_5 \phi_{\max}^4 + 3 b_1^2 b_3 b_8 \phi_{\max}^4 + 3 b_0 b_3^2 b_8 \phi_{\max}^4 \\
&+ b_3^2 b_7 \phi_{\min} \phi_{\max}^4 + 3 b_1 b_3^2 b_8 \phi_{\max}^5 + b_3^3 b_8 \phi_{\max}^6 + 3 b_3^2 b_6 b_8 \phi_{\max}^6 \\
\alpha_1 = b_2^2 &+ 2 b_0 b_2 b_5 + 3 b_0^2 b_2 b_8 + b_2 b_4 \phi_{\min} + 2 b_0 b_2 b_7 \phi_{\min} + b_2 b_6 \phi_{\min}^2 + b_2 b_4 \phi_{\max} + 2 b_1 b_2 b_5 \phi_{\max} + 2 b_0 b_4 b_5 \phi_{\max} \\
&+ 6 b_0 b_1 b_2 b_8 \phi_{\max} + 3 b_0^2 b_4 b_8 \phi_{\max} + b_4^2 \phi_{\min} \phi_{\max} + 2 b_1 b_2 b_7 \phi_{\min} \phi_{\max} + 2 b_0 b_4 b_7 \phi_{\min} \phi_{\max} + b_4 b_6 \phi_{\min}^2 \phi_{\max} \\
&+ 2 b_2 b_3 b_5 \phi_{\max}^2 + 2 b_1 b_4 b_5 \phi_{\max}^2 + b_2 b_6 \phi_{\max}^2 + 2 b_0 b_5 b_6 \phi_{\max}^2 + 3 b_1^2 b_2 b_8 \phi_{\max}^2 + 6 b_0 b_2 b_3 b_8 \phi_{\max}^2 + 6 b_0 b_1 b_4 b_8 \phi_{\max}^2 \\
&+ 3 b_0^2 b_6 b_8 \phi_{\max}^2 + b_4 b_6 \phi_{\min} \phi_{\max}^2 + 2 b_2 b_3 b_7 \phi_{\min} \phi_{\max}^2 + 2 b_1 b_4 b_7 \phi_{\min} \phi_{\max}^2 + 2 b_0 b_6 b_7 \phi_{\min} \phi_{\max}^2 + b_6^2 \phi_{\min}^2 \phi_{\max}^2 \\
&+ 2 b_3 b_4 b_5 \phi_{\max}^3 + 2 b_1 b_5 b_6 \phi_{\max}^3 + 6 b_1 b_2 b_3 b_8 \phi_{\max}^3 + 3 b_1^2 b_4 b_8 \phi_{\max}^3 + 6 b_0 b_3 b_4 b_8 \phi_{\max}^3 + 6 b_0 b_1 b_6 b_8 \phi_{\max}^3 \\
&+ 2 b_3 b_4 b_7 \phi_{\min} \phi_{\max}^3 + 2 b_1 b_6 b_7 \phi_{\min} \phi_{\max}^3 + 2 b_3 b_5 b_6 \phi_{\max}^4 + 3 b_2 b_3^2 b_8 \phi_{\max}^4 + 6 b_1 b_3 b_4 b_8 \phi_{\max}^4 + 3 b_1^2 b_6 b_8 \phi_{\max}^4 \\
&+ 6 b_0 b_3 b_6 b_8 \phi_{\max}^4 + 2 b_3 b_6 b_7 \phi_{\min} \phi_{\max}^4 + 3 b_3^2 b_4 b_8 \phi_{\max}^5 + 6 b_1 b_3 b_6 b_8 \phi_{\max}^5 + 3 b_3^2 b_6 b_8 \phi_{\max}^6
\end{aligned}
$$

$$
\begin{aligned}
\alpha_2 = {} & b_2b_5 + b_2^2b_5 + 2b_0b_5^2 + 3b_0b_2^2b_8 + 3b_0^2b_5b_8 + b_4b_5\phi_{\min} + b_2^2b_7\phi_{\min} + 2b_0b_5b_7\phi_{\min} + b_5b_6\phi_{\min}^2 + 2b_2b_4b_5\phi_{\max} \\
& + 2b_1b_5^2\phi_{\max} + b_2b_7\phi_{\max} + 2b_0b_5b_7\phi_{\max} + 3b_1b_2^2b_8\phi_{\max} + 6b_0b_2b_4b_8\phi_{\max} + 6b_0b_1b_5b_8\phi_{\max} + 3b_0^2b_7b_8\phi_{\max} \\
& + b_4b_7\phi_{\min}\phi_{\max} + 2b_2b_4b_7\phi_{\min}\phi_{\max} + 2b_1b_5b_7\phi_{\min}\phi_{\max} + 2b_0b_7^2\phi_{\min}\phi_{\max} + b_6b_7\phi_{\min}^2\phi_{\max} + b_4^2b_5\phi_{\max}^2 + 2b_3b_5^2\phi_{\max}^2 \\
& + 2b_2b_5b_6\phi_{\max}^2 + 2b_1b_5b_7\phi_{\max}^2 + 3b_2^2b_3b_8\phi_{\max}^2 + 6b_1b_2b_4b_8\phi_{\max}^2 + 3b_0b_4^2b_8\phi_{\max}^2 + 3b_1^2b_5b_8\phi_{\max}^2 + 6b_0b_3b_5b_8\phi_{\max}^2 \\
& + 6b_0b_2b_6b_8\phi_{\max}^2 + 6b_0b_1b_7b_8\phi_{\max}^2 + b_4^2b_7\phi_{\min}\phi_{\max}^2 + 2b_3b_5b_7\phi_{\min}\phi_{\max}^2 + 2b_2b_6b_7\phi_{\min}\phi_{\max}^2 + 2b_1b_7^2\phi_{\min}\phi_{\max}^2 \\
& + 2b_4b_5b_6\phi_{\max}^3 + 2b_3b_5b_7\phi_{\max}^3 + 6b_2b_3b_4b_8\phi_{\max}^3 + 3b_1b_4^2b_8\phi_{\max}^3 + 6b_1b_3b_5b_8\phi_{\max}^3 + 6b_1b_2b_6b_8\phi_{\max}^3 + 6b_0b_4b_6b_8\phi_{\max}^3 \\
& + 3b_1^2b_7b_8\phi_{\max}^3 + 6b_0b_3b_7b_8\phi_{\max}^3 + 2b_4b_6b_7\phi_{\min}\phi_{\max}^3 + 2b_3b_7^2\phi_{\min}\phi_{\max}^3 + b_5b_6^2\phi_{\max}^4 + 3b_3b_4^2b_8\phi_{\max}^4 + 3b_3^2b_5b_8\phi_{\max}^4 \\
& + 6b_2b_3b_6b_8\phi_{\max}^4 + 6b_1b_4b_6b_8\phi_{\max}^4 + 3b_0b_6^2b_8\phi_{\max}^4 + 6b_1b_3b_7b_8\phi_{\max}^4 + b_6^2b_7\phi_{\min}\phi_{\max}^4 + 6b_3b_4b_6b_8\phi_{\max}^5 \\
& + 3b_1b_6^2b_8\phi_{\max}^5 + 3b_3^2b_7b_8\phi_{\max}^5 + 3b_3b_6^2b_8\phi_{\max}^6 \\
\alpha_3 = {} & 2b_2b_5^2 + b_2b_8 + b_2^3b_8 + 2b_0b_5b_8 + 6b_0b_2b_5b_8 + 3b_0^2b_8^2 + 2b_2b_5b_7\phi_{\min} + b_4b_8\phi_{\min} + 2b_0b_7b_8\phi_{\min} + b_6b_8\phi_{\min}^2 \\
& + 2b_4b_5^2\phi_{\max} + 2b_2b_5b_7\phi_{\max} + 3b_2^2b_4b_8\phi_{\max} + 2b_1b_5b_8\phi_{\max} + 6b_1b_2b_5b_8\phi_{\max} + 6b_0b_4b_5b_8\phi_{\max} + 6b_0b_2b_7b_8\phi_{\max} \\
& + 6b_0b_1b_8^2\phi_{\max} + 2b_4b_5b_7\phi_{\min}\phi_{\max} + 2b_2b_7^2\phi_{\min}\phi_{\max} + 2b_1b_7b_8\phi_{\min}\phi_{\max} + 2b_5^2b_6\phi_{\max}^2 + 2b_4b_5b_7\phi_{\max}^2 + 3b_2b_4^2b_8\phi_{\max}^2 \\
& + 2b_3b_5b_8\phi_{\max}^2 + 6b_2b_3b_5b_8\phi_{\max}^2 + 6b_1b_4b_5b_8\phi_{\max}^2 + 3b_2^2b_6b_8\phi_{\max}^2 + 6b_0b_5b_6b_8\phi_{\max}^2 + 6b_1b_2b_7b_8\phi_{\max}^2 + 6b_0b_4b_7b_8\phi_{\max}^2 \\
& + 3b_1^2b_8^2\phi_{\max}^2 + 6b_0b_3b_8^2\phi_{\max}^2 + 2b_5b_6b_7\phi_{\min}\phi_{\max}^2 + 2b_4b_7^2\phi_{\min}\phi_{\max}^2 + 2b_3b_7b_8\phi_{\min}\phi_{\max}^2 + 2b_5b_6b_7p_1^3 + b_4^3b_8\phi_{\max}^3 \\
& + 6b_3b_4b_5b_8\phi_{\max}^3 + 6b_2b_4b_6b_8\phi_{\max}^3 + 6b_1b_5b_6b_8\phi_{\max}^3 + 6b_2b_3b_7b_8\phi_{\max}^3 + 6b_1b_4b_7b_8\phi_{\max}^3 + 6b_0b_6b_7b_8\phi_{\max}^3 \\
& + 6b_1b_3b_8^2\phi_{\max}^3 + 2b_6b_7^2\phi_{\min}\phi_{\max}^3 + 3b_4^2b_6b_8\phi_{\max}^4 + 6b_3b_5b_6b_8\phi_{\max}^4 + 3b_2b_6^2b_8\phi_{\max}^4 + 6b_3b_4b_7b_8\phi_{\max}^4 \\
& + 6b_1b_6b_7b_8\phi_{\max}^4 + 3b_3^2b_8^2\phi_{\max}^4 + 3b_4b_6^2b_8\phi_{\max}^5 + 6b_3b_6b_7b_8\phi_{\max}^5 + b_6^3b_8\phi_{\max}^6 \\
\alpha_4 = {} & b_5^3 + 2b_2b_5b_8 + 3b_2^2b_5b_8 + 3b_0b_5^2b_8 + 6b_0b_2b_8^2 + b_5^2b_7\phi_{\min} + 2b_2b_7b_8\phi_{\min} + 2b_5^2b_7\phi_{\max} + 2b_4b_5b_8\phi_{\max} \\
& + 6b_2b_4b_5b_8\phi_{\max} + 3b_1b_5^2b_8\phi_{\max} + 3b_2^2b_7b_8\phi_{\max} + 6b_0b_5b_7b_8\phi_{\max} + 6b_1b_2b_8^2\phi_{\max} + 6b_0b_4b_8^2\phi_{\max} + 2b_5b_7^2\phi_{\min}\phi_{\max} \\
& + 2b_4b_7b_8\phi_{\min}\phi_{\max} + b_5b_7^2\phi_{\max}^2 + 3b_4^2b_5b_8\phi_{\max}^2 + 3b_3b_5^2b_8\phi_{\max}^2 + 2b_5b_6b_8\phi_{\max}^2 + 6b_2b_5b_6b_8\phi_{\max}^2 + 6b_2b_4b_7b_8\phi_{\max}^2 \\
& + 6b_1b_5b_7b_8\phi_{\max}^2 + 3b_0b_7^2b_8\phi_{\max}^2 + 6b_2b_3b_8^2\phi_{\max}^2 + 6b_1b_4b_8^2\phi_{\max}^2 + 6b_0b_6b_8^2\phi_{\max}^2 + b_7^3\phi_{\min}\phi_{\max}^2 + 2b_6b_7b_8\phi_{\min}\phi_{\max}^2 \\
& + 6b_4b_5b_6b_8\phi_{\max}^3 + 3b_4^2b_7b_8\phi_{\max}^3 + 6b_3b_5b_7b_8\phi_{\max}^3 + 6b_2b_6b_7b_8\phi_{\max}^3 + 3b_1b_7^2b_8\phi_{\max}^3 + 6b_3b_4b_8^2\phi_{\max}^3 + 6b_1b_6b_8^2\phi_{\max}^3 \\
& + 3b_5b_6^2b_8\phi_{\max}^4 + 6b_4b_6b_7b_8\phi_{\max}^4 + 3b_3b_7^2b_8\phi_{\max}^4 + 6b_3b_6b_8^2\phi_{\max}^4 + 3b_6^2b_7b_8\phi_{\max}^5 \\
\alpha_5 = {} & 2b_5^2b_8 + 3b_2b_5^2b_8 + 3b_2^2b_8^2 + 6b_0b_5b_8^2 + 2b_5b_7b_8\phi_{\min} + 3b_4b_5^2b_8\phi_{\max} + 2b_5b_7b_8\phi_{\max} + 6b_2b_5b_7b_8\phi_{\max} + 6b_2b_4b_8^2\phi_{\max} \\
& + 6b_1b_5b_8^2\phi_{\max} + 6b_0b_7b_8^2\phi_{\max} + 2b_7^2b_8\phi_{\min}\phi_{\max} + 3b_5^2b_6b_8\phi_{\max}^2 + 6b_4b_5b_7b_8\phi_{\max}^2 + 3b_2b_7^2b_8\phi_{\max}^2 + 3b_4^2b_8^2\phi_{\max}^2 \\
& + 6b_3b_5b_8^2\phi_{\max}^2 + 6b_2b_6b_8^2\phi_{\max}^2 + 6b_1b_7b_8^2\phi_{\max}^2 + 6b_5b_6b_7b_8\phi_{\max}^3 + 3b_4b_7^2b_8\phi_{\max}^3 + 6b_4b_6b_8^2\phi_{\max}^3 + 6b_3b_7b_8^2\phi_{\max}^3 \\
& + 3b_6b_7^2b_8\phi_{\max}^4 + 3b_6^2b_8^2\phi_{\max}^4\alpha_6 \qquad = b_5^3b_8 \\
& + 3b_5b_7^2b_8\phi_{\max}^2 + 6b_5b_6b_8^2\phi_{\max}^2 + 6b_4b_7b_8^2\phi_{\max}^2 + 3b_3b_8^3\phi_{\max}^2 + b_7^3b_8\phi_{\max}^3 + 6b_6b_7b_8^2\phi_{\max}^3 \\
\alpha_7 = {} & 3b_5^2b_8^2 + 3b_2b_8^3 + 6b_5b_7b_8^2\phi_{\max} + 3b_4b_8^3\phi_{\max} + 3b_7^2b_8^2\phi_{\max}^2 + 3b_6b_8^3\phi_{\max}^2 \\
\alpha_8 = {} & 3b_5b_8^3 + 3b_7b_8^3\phi_{\max} \\
\alpha_9 = {} & b_8^4
\end{aligned}
$$